

# The structure of Rényi entropic inequalities

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We investigate the universal inequalities relating the  $\alpha$ -Rényi entropies of the marginals of a multi-partite quantum state. This is in analogy to the same question for the Shannon and von Neumann entropy ( $\alpha = 1$ ) which are known to satisfy several non-trivial inequalities such as strong subadditivity. Somewhat surprisingly, we find for  $0 < \alpha < 1$ , that the only inequality is non-negativity: In other words, any collection of non-negative numbers assigned to the nonempty subsets of  $n$  parties can be arbitrarily well approximated by the  $\alpha$ -entropies of the  $2^n - 1$  marginals of a quantum state.

For  $\alpha > 1$  we show analogously that there are no non-trivial *homogeneous* (in particular no linear) inequalities. On the other hand, it is known that there are further, non-linear and indeed non-homogeneous, inequalities delimiting the  $\alpha$ -entropies of a general quantum state.

Finally, we also treat the case of Rényi entropies restricted to classical states (i.e. probability distributions), which in addition to non-negativity are also subject to monotonicity. For  $\alpha \neq 0, 1$  we show that this is the only other homogeneous relation.

## I. PROLOGUE

The von Neumann entropy  $S(\rho) = -\text{Tr} \rho \log \rho$  of a quantum state  $\rho$  is a key notion in quantum information theory [43] as well as in statistical physics [31]. It is furthermore the canonical measure of entanglement for bipartite pure states [6]. In many cases the relative magnitude of the entropy of the reduced states of different subsystems is important. Thus for example for a tripartite state  $\rho_{ABC}$  one can compute  $S(\rho_A)$ ,  $S(\rho_B)$ , and so on. For any positive number  $a$  one can find a quantum state such that  $S(\rho_A) = a$ , for example. However for a fixed quantum state, there are inequalities between the values of the entropies of the reduced states of the subsystems.

There are essentially two such unconstrained inequalities known (up to permutation of the parties), *strong subadditivity* and *weak monotonicity* [20, 37]:

$$\begin{aligned} S(\rho_{ABC}) + S(\rho_B) &\leq S(\rho_{AB}) + S(\rho_{BC}), \\ S(\rho_A) + S(\rho_B) &\leq S(\rho_{AC}) + S(\rho_{BC}). \end{aligned} \tag{1}$$

There are no other constraints for up to 3 parties, but the analogous statement is a major open problem for larger  $n$  [8, 21, 37]. Indeed we anticipate that the question may be very complicated in general. For example the analogous question for classical (Shannon) entropies has been much studied and in this case, for  $n \geq 4$  there are infinitely many independent (linear) inequalities known [24, 45, 46]. All these inequalities, and more discovered by Makarychev *et al.* [23] and Dougherty *et al.* [11] might very well hold for the von Neumann entropy, too. Indeed it is known that the von Neumann entropy satisfies some constrained inequalities that are counterparts of known classical constrained inequalities [8].

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For a state of  $n$  parties there are  $2^n - 1$  non-trivial von Neumann entropies, one corresponding to each non-empty subset of parties. Thus the existence of these inequalities means that given a set of  $2^n - 1$  positive numbers there will, in general, be no quantum state whose reduced state entropies have these values.

In this paper we consider the analogous questions for quantum Rényi entropies, also called  $\alpha$ -entropies [38], of a quantum state  $\rho$  (given as a unit trace density operator on a suitable Hilbert space):

$$S_\alpha(\rho) = \frac{1}{1-\alpha} \log \text{Tr} \rho^\alpha, \quad (2)$$

for  $0 \leq \alpha \leq \infty$ . (We note that the case of  $\alpha = 1$  is the von Neumann entropy).

We will show that, very surprisingly, the case of Rényi entropies for  $\alpha \neq 1$  is much different from that for the von Neumann entropy:

For  $0 < \alpha < 1$ , we will show (in theorem 3) that the only inequality is non-negativity  $S_\alpha(\rho) \geq 0$ . In other words, any collection of non-negative numbers assigned to the nonempty subsets of  $n$  parties can be arbitrarily well approximated by the  $\alpha$ -entropies of the  $2^n - 1$  marginals of a quantum state.

For  $\alpha > 1$  we will prove that there are no linear (or indeed homogeneous) inequalities. We show (in theorem 7) that given any vector  $\mathbf{v}$  of  $2^n - 1$  positive numbers, it may or may not be the case that this is the vector of the  $2^n - 1$  marginals of a quantum state; however it is arbitrarily well approximated by a positive multiple of the  $\alpha$ -entropies of the  $2^n - 1$  marginals of a quantum state. On the other hand (contrary to the case  $0 < \alpha < 1$ ) there are other, nonlinear, inequalities delimiting the set of possible entropy vectors; one such inequality was proved in [4], which we recall in section IV.

Finally, we show (in section V) that in the classical case the only homogeneous inequalities are non-negativity and monotonicity (under the inclusion of subsets of parties), for all  $\alpha \neq 0, 1$ .

## II. THE RÉNYI ENTROPY

The definition in (2) is clearly well-defined, and continuous in the state as well as in  $\alpha$ , for  $\alpha \in (0, \infty) \setminus \{1\}$ . For  $\alpha = 0, 1, \infty$ , the function is defined by taking a limit, yielding

$$\begin{aligned} S_0(\rho) &= \log \text{rank } \rho, \\ S_1(\rho) &= S(\rho) := -\text{Tr} \rho \log \rho \quad (\text{von Neumann entropy}), \\ S_\infty(\rho) &= -\log \|\rho\|, \end{aligned}$$

where  $\|\cdot\|$  denotes the operator norm, i.e.  $\|\rho\|$  is the largest eigenvalue of  $\rho$ .

By their definition, all of the quantum Rényi entropies depend only on the spectrum of  $\rho$ , which we can think of as a probability distribution  $P$ . In this sense, the above formulas generalise the notion

$$H_\alpha(P) = \frac{1}{1-\alpha} \log \left( \sum_x P(x)^\alpha \right)$$

introduced by Rényi in his axiomatic investigation of information measures for random variables and their distributions [38], following Shannon's example [39]. This approach has generated a lot of subsequent activity [1].

It is easy to see that for states  $\rho \geq 0$ ,  $\text{Tr} \rho = 1$ , all  $S_\alpha(\rho) \geq 0$ , with equality if and only if  $\rho$  is pure, i.e. a rank-one projector  $\rho = |\varphi\rangle\langle\varphi|$ . Furthermore, for fixed  $\rho$ , the function  $\alpha \mapsto S_\alpha(\rho)$  is monotonically non-increasing [38].

Many other useful, interesting and curious mathematical properties of the Rényi entropies are known [1].

Rényi entropies, and, more generally, Rényi relative entropies and the corresponding channel capacities play an important role in classical as well as quantum information theory. The Rényi quantities with parameter  $\alpha \in (0, 1)$  are related to the so-called direct domain of information theoretic problems. They can be used to quantify the trade-off between the rates of the two types of error probabilities in binary state discrimination [5, 10, 17, 18, 30], which in turn yields a trade-off relation between the error rate and the compression rate in state compression (see [10] for the classical case; the quantum case is completely analogous). The related capacities quantify the trade-off between the error rate and the coding rate for classical information transmission [10, 17], and can be used to obtain lower bounds on the single-shot classical capacities [28]. The Rényi quantities with parameter  $\alpha > 1$  are related to converse problems. They can be used to quantify the trade-off between the rates of the type I error and the type II success probability in binary state discrimination [10, 16, 34], as well as the trade-off between the rate of the success probability and the compression rate in state compression [10]. The related capacity formulas give bounds on the success rate for coding rates above the Holevo capacity [10, 22, 34], and can be used to give upper bounds on the single-shot classical capacities of quantum channels [29]. Also, Rényi entropies feature prominently in the theory of bipartite pure state transformations by local operations and classical communication: Only recently it was shown [3] that the monotonicity of the Rényi entropies of the reduced states for  $\alpha > 1$  is both necessary and sufficient for catalytic transformations (whereas unassisted transformations are long known to be characterized by majorization [13, 32]). And in [14], Rényi entropies (essentially  $\alpha = 0$  and  $\alpha = \infty$ ) were used to put bounds on the classical communication required for a given transformation. And finally, Rényi entropies were employed to put lower bounds on the communication complexity of certain distributed computation problems [25, 42].

While the von Neumann entropy can be obtained as the limit of the Rényi entropies for  $\alpha \rightarrow 1$ , and hence it can be considered as one particular member of this parametric family of entropies, its basic properties sharply distinguish it from all other members of the family. Indeed, while the von Neumann entropy is strongly subadditive, the other Rényi entropies with  $\alpha \in (0, +\infty) \setminus \{1\}$  are not even subadditive.

To illustrate the consequences of this difference, we mention the problem of entropy asymptotics on spin chains. Given a translation-invariant state  $\rho$  on an infinite spin chain, subadditivity of entropy ensures the existence of the limit  $s(\rho) := \lim_{n \rightarrow +\infty} \frac{1}{n} S(\rho_{[1,n]})$ , where  $\rho_{[1,n]}$  is the restriction of  $\rho$  to any  $n$  consecutive sites, and  $s(\rho)$  gives the ultimate compression rate for an ergodic  $\rho$  [7]. More refined knowledge about the decay of error for rates below  $s(\rho)$  can be obtained using the method developed in [18]; for this, however, one has to show the existence of the regularized Rényi entropies  $s_\alpha(\rho) := \lim_{n \rightarrow +\infty} \frac{1}{n} S_\alpha(\rho_{[1,n]})$  for every  $\alpha \in (0, 1)$ . Due to the lack of subadditivity, the existence of this limit is not at all straightforward, and is actually only known for some special classes of states [18, 26, 27].

When  $\rho$  is pure, the block entropies  $S_\alpha(\rho_{[1,n]})$  are used to measure the entanglement between the block  $[1, n]$  and the rest of the chain, and the scaling of these entropies are closely related to the presence or absence of criticality in the system [12, 41]. It follows from strong subadditivity that the entanglement entropy  $S(\rho_{[1,n]})$  is a monotone increasing function of the block size [2]. This is no longer true when the entanglement is measured by some Rényi entropy; a counterexample with oscillating block Rényi entropies for  $\alpha > 2$  was found in [15]. It is not known, however, whether such oscillating behaviour can happen for Rényi entropies with parameter  $\alpha$  arbitrarily close to 1.

In the view of the above examples, it is natural to ask whether there are other universal inequalities between the Rényi entropies of the subsystems of a multipartite quantum system, and

this is what we are going to investigate in the following.

To fix notation, we shall concern ourselves with  $n$ -partite quantum systems with generic tensor product Hilbert space  $\mathcal{H} = \mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_n$ . Within the discussion we usually consider  $n$  and  $\alpha$  to be fixed, but the local systems are unconstrained, i.e. we do not impose limits on the dimension of the  $\mathcal{H}_i$ . For a state  $\rho$  on  $\mathcal{H}$  we have the reduced states  $\rho_I = \text{Tr}_{I^c} \rho$ , with the partial trace over all parties in the complement  $I^c = [n] \setminus I$ , and we shall consider them and their entropies all at once, for all non-empty subsets of  $[n] = \{1, \dots, n\}$ . The power set and the power set without the empty set we denote as follows:

$$\begin{aligned}\mathcal{P}[n] &:= \{I \subset [n]\}, \\ \mathcal{P}_\emptyset[n] &:= \{I \subset [n] : I \neq \emptyset\} = \mathcal{P}[n] \setminus \{\emptyset\}.\end{aligned}$$

We are interested in the universal relations obeyed by the  $\alpha$ -entropies of a general  $n$ -party state  $\rho$ . For instance, by definition clearly

$$S_\alpha(\rho_I) \geq 0$$

for all subsets  $I \subset [n]$ .

Note that via the usual diagonal matrix representation we can view a probability distribution of  $n$  discrete random variables as a quantum state, and conversely, states which are diagonal in a tensor product basis of an  $n$ -party system can be identified with a classical  $n$ -party probability distribution, and hence we will call states of that form *classical*. In this case, there is another inequality,

$$S_\alpha(\rho_I) \leq S_\alpha(\rho_J)$$

for  $I \subset J$ , i.e. monotonicity of the entropy function with respect to subset inclusion.

These examples motivate the introduction of the set of all entropic vectors,

$$\Sigma_\alpha^n := \{(S_\alpha(\rho_I))_{I \in \mathcal{P}_\emptyset[n]} : \rho \text{ state}\} \subset \mathbb{R}^{\mathcal{P}_\emptyset[n]},$$

and the same for classical case

$$\begin{aligned}\Gamma_\alpha^n &:= \{(S_\alpha(\rho_I))_{I \in \mathcal{P}_\emptyset[n]} : \rho \text{ classical state}\} \\ &= \{(H_\alpha(P_I))_{I \in \mathcal{P}_\emptyset[n]} : P \text{ prob. distr.}\} \subset \mathbb{R}^{\mathcal{P}_\emptyset[n]}.\end{aligned}$$

In fact, as we tend to consider “ $\leq$ ” type inequalities between continuous functions of the coordinates, it makes sense to focus on the topological closures  $\overline{\Sigma}_\alpha^n, \overline{\Gamma}_\alpha^n \subset \mathbb{R}^{\mathcal{P}_\emptyset[n]}$ . Those universal inequalities we are looking for are the constraints describing the geometric shape of these sets.

The above examples of known inequalities are homogeneous, indeed linear, relations. (By a homogeneous inequality we mean an inequality of the form  $f(\mathbf{v}) \geq 0$ ,  $\mathbf{v} \in \Sigma_\alpha^n$ , where  $f$  is a homogeneous function on  $\mathbb{R}_{\geq 0}^{\mathcal{P}_\emptyset[n]}$ , i.e., there exists a  $d \in \mathbb{R}$  such that  $f(\lambda \mathbf{v}) = \lambda^d f(\mathbf{v})$  holds for every  $\lambda \in \mathbb{R}_{\geq 0}$  and  $\mathbf{v} \in \mathbb{R}_{\geq 0}^{\mathcal{P}_\emptyset[n]}$ .) That it is meaningful to look for such relations is motivated by the observation that all Rényi entropies are *extensive*, i.e.

$$S_\alpha(\rho \otimes \sigma) = S_\alpha(\rho) + S_\alpha(\sigma).$$

And since this is true for all subset reduced states simultaneously, we have for non-negative integers  $k$  and  $\ell$ ,

$$k\Sigma_\alpha^n + \ell\Sigma_\alpha^n \subset \Sigma_\alpha^n, \quad k\Gamma_\alpha^n + \ell\Gamma_\alpha^n \subset \Gamma_\alpha^n,$$

and likewise for the respective closures. If this held for non-negative *reals* it would mean that the corresponding set is a convex cone. This is indeed known for  $\alpha = 1$  [37, 44], but not true for  $\alpha > 1$  (see below).

### III. $0 < \alpha < 1$

In this section,  $\alpha$  is fixed in the interval  $(0, 1)$ . We start off with a simple classical construction. For  $I \in \mathcal{P}_0[n]$ , let  $\delta_I$  denote the corresponding basis vector in  $\mathbb{R}^{\mathcal{P}_0[n]}$ , i.e.,  $\delta_I$  is the characteristic function of the singleton  $\{I\}$ .

**Lemma 1** *For any  $s > 0$ , the vector  $s\delta_{[n]} \in \mathbb{R}_{\geq 0}^{\mathcal{P}_0[n]}$  is approximately  $\alpha$ -entropic, i.e.  $s\delta_{[n]} \in \overline{\Sigma}_\alpha^n$ . In fact, this vector can be approximated arbitrarily by classical states.*

**Proof** For integers  $M_1, \dots, M_n$  consider “local” alphabets  $\mathcal{X}_i := \{0\} \cup [M_i]$  and define distributions  $P_{t; \{M_i\}}(0 \leq t \leq 1)$  on the Cartesian product  $\mathcal{X}_1 \times \dots \times \mathcal{X}_n$  as follows:

$$P_{t; \{M_i\}}(x_1, \dots, x_n) := \begin{cases} 1-t & \text{if } x_1 = \dots = x_n = 0, \\ \frac{t}{M_1 \dots M_n} & \text{if } x_1, \dots, x_n \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The marginals on  $\mathcal{X}_I = \times_{i \in I} \mathcal{X}_i$ , for a subset  $I \subset [n]$ , are easy to construct: they are given precisely by  $P_{t; \{M_i; i \in I\}}(x_i : i \in I)$ . The corresponding quantum state and its marginals hence are

$$\begin{aligned} \rho &= \sum_{x_1, \dots, x_n} P_{t; \{M_i\}}(x_1, \dots, x_n) |x_1\rangle\langle x_1| \otimes \dots \otimes |x_n\rangle\langle x_n|, \\ \rho_I &= \sum_{x_i; i \in I} P_{t; \{M_i; i \in I\}}(x_i : i \in I) \bigotimes_{i \in I} |x_i\rangle\langle x_i|. \end{aligned}$$

With this, the Rényi entropies are straightforward to compute:

$$S_\alpha(\rho_I) = \frac{1}{1-\alpha} \log \left( (1-t)^\alpha + t^\alpha M_I^{1-\alpha} \right),$$

with  $M_I = \prod_{i \in I} M_i$ .

Now, for sufficiently large  $M_{[n]}$ , we can set

$$t := \left( \frac{2^{s(1-\alpha)} - 1}{M_{[n]}^{1-\alpha}} \right)^{\frac{1}{\alpha}}. \quad (3)$$

Then, in the limit  $\min\{M_1, \dots, M_n\} \rightarrow \infty$ ,

$$S_\alpha(\rho_{[n]}) = \frac{1}{1-\alpha} \log \left( (1-t)^\alpha + 2^{s(1-\alpha)} - 1 \right) \rightarrow s,$$

since  $t \rightarrow 0$ . On the other hand, for  $I \subsetneq [n]$ ,

$$S_\alpha(\rho_I) = \frac{1}{1-\alpha} \log \left( (1-t)^\alpha + \frac{2^{s(1-\alpha)} - 1}{M_{[n] \setminus I}^{1-\alpha}} \right) \rightarrow 0,$$

because  $M_i \rightarrow \infty$ . □

**Proposition 2** *For any  $\emptyset \neq I \subset [n]$ , and  $s > 0$ , the vector  $s\delta_I \in \mathbb{R}_{\geq 0}^{\mathcal{P}_0[n]}$  is approximately  $\alpha$ -entropic, i.e.  $s\delta_I \in \overline{\Sigma}_\alpha^n$ .*

**Proof** It is enough to show that for any  $\epsilon > 0$ , there exist local systems  $\mathcal{H}_1, \dots, \mathcal{H}_n, \mathcal{H}_{n+1}$  and a pure state  $\rho = |\psi\rangle\langle\psi|$  on  $\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_n \otimes \mathcal{H}_{n+1}$  with

$$s - \epsilon \leq S_\alpha(\rho_I) \leq s + \epsilon, \quad \text{and} \quad S_\alpha(\rho_J) \leq \epsilon \quad \text{if } J \subset [n], J \neq I.$$

If  $I = [n]$ , we just use the  $(n+1)$ st party to purify the state. If  $|I| = 1$ , we likewise take the classical state of lemma 1 on the  $n$ -party system  $[n+1] \setminus I$  and purify it using the system  $I$ . Thus, from now on we may assume that  $k = |I|$  and  $\ell = |I^c| = n+1-k$  are both  $\geq 2$ . The idea is that for integer  $M$ , the distributions  $P_{t; \{M^\ell: i \in I\}}$  on the systems  $I$ , and  $P_{t; \{M^k: j \in I^c\}}$  on the systems  $I^c = [n+1] \setminus I$ , both with the same  $t$  given by

$$t = \left( \frac{2^{s(1-\alpha)} - 1}{M^{k\ell(1-\alpha)}} \right)^{\frac{1}{\alpha}},$$

have the same nonzero probabilities, just arranged differently. In other words, the corresponding classical states are isospectral, hence we may view them as reduced states of an  $(n+1)$ -party pure state.

In detail, we may without loss of generality relabel the systems such that  $I = \{1, \dots, k\}$  and  $I^c = \{k+1, \dots, k+\ell = n+1\}$ . For every  $i \in \{1, \dots, k\}$  and  $j \in \{1, \dots, \ell\}$ , let  $\mathcal{K}_{ij}$  be an  $M$ -dimensional Hilbert space with an orthonormal basis  $\{|k\rangle_{ij} : k = 1, \dots, M\}$ , and define

$$\mathcal{H}_i := \mathbb{C}|0\rangle_i \oplus \left( \bigotimes_{j=1}^{\ell} \mathcal{K}_{ij} \right), \quad \mathcal{H}_{k+j} := \mathbb{C}|0\rangle_{k+j} \oplus \left( \bigotimes_{i=1}^k \mathcal{K}_{ij} \right),$$

where  $|0\rangle_i$  are unit vectors. For a  $k \times \ell$  matrix  $x \in [M]^{k \times \ell}$  and  $i \in \{1, \dots, k\}$ ,  $j \in \{1, \dots, \ell\}$ , let

$$|x^{(i)}\rangle := \bigotimes_{j=1}^{\ell} |x_j^{(i)}\rangle_{ij} \in \mathcal{H}_i, \quad |x_{(k+j)}\rangle := \bigotimes_{i=1}^k |x_j^{(i)}\rangle_{ij} \in \mathcal{H}_{k+j},$$

and

$$|\psi\rangle := \sqrt{1-t} \bigotimes_{i=1}^{n+1} |0\rangle_i + \sqrt{\frac{t}{M^{k\ell}}} \sum_{x \in [M]^{k \times \ell}} |x^{(1)}\rangle \dots |x^{(k)}\rangle |x_{(k+1)}\rangle \dots |x_{(n+1)}\rangle.$$

(Here  $|x\rangle|y\rangle$  stands for  $|x\rangle \otimes |y\rangle$ .)

Let  $\rho := \rho_{[n]} = \text{Tr}_{n+1} |\psi\rangle\langle\psi|$ . The crucial property of this definition is that every party  $i \in I$  and  $j \in I^c$  have a coordinate in common, namely  $x_j^{(i)} \in [M]$ . One can easily see that  $\rho_I$  is a classical state of the type studied in lemma 1, and the same calculation as in lemma 1 shows that

$$S_\alpha(\rho_I) = \frac{1}{1-\alpha} \log \left( (1-t)^\alpha + 2^{s(1-\alpha)} - 1 \right) \rightarrow s,$$

as  $M \rightarrow \infty$ . If  $J \in \mathcal{P}_\emptyset[n]$  is different from  $I$  then there are  $i \in I$  and  $j \in I^c = [n+1] \setminus I$ , such that either  $i, j \in J$  or  $i, j \in J^c$ . The second case has entropy equivalent to the first since we may just go to the complementary set. In the first case, we have

$$\rho_J = (1-t) \bigotimes_{i \in J} |0\rangle_i \langle 0|_i + t\sigma,$$

where  $\sigma$  is supported on a space which contains each  $\mathcal{K}_{ij}$  at most once if the  $i$ th or the  $(k+j)$ th system has been traced out, and twice otherwise. Hence,

$$\sigma = \left( \bigotimes_{i \in J \cap I} \bigotimes_{j \in J \cap [n] \setminus I} |\psi_{ij}\rangle\langle\psi_{ij}| \right) \otimes \sigma',$$

where

$$|\psi_{ij}\rangle = \frac{1}{\sqrt{M}} \sum_{k=1}^M |k\rangle_{ij} |k\rangle_{ij} \in \mathcal{K}_{ij} \otimes \mathcal{K}_{ij},$$

and  $\sigma'$  is a density operator supported on a space of dimension at most  $M^{k\ell-1}$ . Hence,  $S_\alpha(\sigma) = S_\alpha(\sigma') \leq \log M^{k\ell-1}$ , or equivalently,  $\text{Tr } \sigma^\alpha \leq M^{(k\ell-1)(1-\alpha)}$ . This yields

$$\begin{aligned} S_\alpha(\rho_J) &\leq \frac{1}{1-\alpha} \log \left( (1-t)^\alpha + M^{(k\ell-1)(1-\alpha)} t^\alpha \right) \\ &= \frac{1}{1-\alpha} \log \left( (1-t)^\alpha + \frac{1}{M^{1-\alpha}} (2^{s(1-\alpha)} - 1) \right) \rightarrow 0, \end{aligned}$$

as  $M \rightarrow \infty$ .  $\square$

**Theorem 3** Every element  $v \in \mathbb{R}_{\geq 0}^{\mathcal{P}_0[n]}$  is approximately  $\alpha$ -entropic. In other words, there are no non-trivial inequalities constraining the Rényi entropies (with fixed  $\alpha < 1$ ) of a multi-party state: The only restriction is non-negativity:

$$\overline{\Sigma}_\alpha^n = \mathbb{R}_{\geq 0}^{\mathcal{P}_0[n]}.$$

**Proof** Via proposition 2 this is quite obvious: Observe  $v = \sum_{I \in \mathcal{P}_0[n]} v_I \delta_I$ , and that for each subset  $I \subset [n]$  we can find an  $n$ -party state  $\rho^{(I)}$  such that its entropies arbitrarily approximate  $v_I \delta_I$ , i.e.

$$|S_\alpha(\rho_J^{(I)}) - v_I \delta_{I,J}| \leq \epsilon \text{ for all } I, J \subset [n].$$

Letting  $\rho := \bigotimes_{I \in \mathcal{P}_0[n]} \rho^{(I)}$ , we are done, since  $\epsilon$  can be chosen arbitrarily small.  $\square$

**Remark 4** From theorem 3 we can see that  $\Sigma_\alpha^n$  is not a closed set (assuming  $n \geq 2$ ). Indeed, we found that for any  $I \subset [n]$ , the ray  $\mathbb{R}_{\geq 0} \delta_I$  is in  $\overline{\Sigma}_\alpha^n$ . However, it is easy to see that except for the origin, none of its points  $s \delta_I$  can be an element of  $\Sigma_\alpha^n$ .

For otherwise there would be a state  $\rho$  with  $S_\alpha(\rho_I) = s$  and all other  $S_\alpha(\rho_J) = 0$ . Now, if  $|I| \geq 2$ , say  $I = \{i_1, i_2, \dots\}$ , then  $S_\alpha(\rho_i) = 0$  implies that all single-party marginals  $\rho_i$  are pure, meaning that  $\rho = \rho_1 \otimes \dots \otimes \rho_n$  is pure, too. Hence we would have  $S(\rho_I) = 0$  as well. If on the other hand  $I = \{i\}$ , then we may choose  $j \notin I$  and reason similarly that  $\rho_j$  is a pure state, hence  $\rho_{\{i,j\}} = \rho_i \otimes \rho_j$  and so  $S(\rho_{\{i,j\}}) = S(\rho_i) + S(\rho_j) = s \neq 0$ , obtaining a contradiction again.

#### IV. $1 < \alpha \leq \infty$

As in the previous section, we start with the basic construction to attain entropy vectors arbitrarily close to the coordinate axes. Throughout the section,  $1 < \alpha \leq \infty$  is fixed.

**Lemma 5** For all  $s > 0$ , there is a vector  $s \delta_{[n]} + O(1) \in \Sigma_\alpha^n$ . To be precise, there exists a constant  $C$ , which may be chosen as  $C = \frac{1}{1-\frac{1}{\alpha}} \log n$ , and classical states with

$$s \leq S_\alpha(\rho) \leq s + C, \quad \text{and} \quad S_\alpha(\rho_J) \leq C \text{ for } J \neq [n].$$

In particular,  $\delta_{[n]} \in \overline{\mathbb{R}_{\geq 0} \Sigma_\alpha^n}$ .

**Proof** The following argument is presented for  $\alpha < \infty$ ; to obtain the claims in the case  $\alpha = \infty$  one simply takes the limit.

For an integer  $M$  consider “local” alphabets  $\mathcal{X}_i := \{0\} \cup [M]$  and define distributions  $\mathcal{Q}_{[n]:R}$  on the Cartesian product  $\mathcal{X}_1 \times \cdots \times \mathcal{X}_n$  as follows:

$$\mathcal{Q}_{[n]:R}(x_1, \dots, x_n) := \begin{cases} \frac{1}{n} R(x_i) & \text{if } x_i \neq 0 \text{ and } x_j = 0 \forall j \neq i, \\ 0 & \text{otherwise,} \end{cases}$$

where  $R$  is an arbitrary probability distribution on  $[M]$ .

For the corresponding classical state, it is straightforward to verify that

$$S_\alpha(\rho_{[n]}) = \frac{1}{1-\alpha} \log \left( n \sum_{x=1}^M \left( \frac{1}{n} R(x) \right)^\alpha \right) = \log n + H_\alpha(R).$$

On the other hand, the marginal state  $\rho_I$  for any  $I \subsetneq [n]$  has an eigenvalue  $\lambda \geq \frac{1}{n}$ , hence

$$S_\alpha(\rho_I) \leq \frac{1}{1-\alpha} \log \lambda^\alpha \leq \frac{1}{1-\alpha} \log \left( \frac{1}{n} \right)^\alpha = C,$$

and we are done, since we can choose  $M$  large enough to accommodate a distribution  $R$  on  $[M]$  with  $H_\alpha(R) = s$ .  $\square$

**Proposition 6** For all  $s > 0$  and  $I \in \mathcal{P}_0[n]$ , there is a vector  $s\delta_I + O(1) \in \Sigma_\alpha^n$ . To be precise, there exists a constant  $C$ , which may be chosen as  $C = \frac{1}{1-\alpha} \log(|I|(n+1-|I|))$ , and states with

$$s \leq S_\alpha(\rho_I) \leq s + C, \quad \text{and} \quad S_\alpha(\rho_J) \leq C \text{ for } J \neq I.$$

In particular,  $\delta_I \in \overline{\mathbb{R}_{\geq 0} \Sigma_\alpha^n}$ .

**Proof** If  $I = [n]$ , this is lemma 5, but we shall present a direct quantum construction, of a pure state on  $n+1$  parties; hence view  $I$  as a subset of  $[n+1]$ . Pick a distribution  $R$  on some finite alphabet  $[M]$  with  $H_\alpha(R) = s$  and fix a purification of  $R$  – understood as a quantum state  $R = \sum_x R(x) |x\rangle\langle x|$  –,  $|\mu\rangle = \sum_{x=1}^M \sqrt{R(x)} |x\rangle |x\rangle$ .

Now, construct the following  $(n+1)$ -party pure state vector,

$$|\psi\rangle := \sqrt{\frac{1}{|I||I^c|}} \bigoplus_{i \in I, j \in I^c} |\mu\rangle_{ij} \otimes |ij\rangle^{\otimes [n]},$$

where  $I^c = [n+1] \setminus I$ , and  $\rho := \text{Tr}_{n+1} |\psi\rangle\langle\psi|$ . However, our reasoning will be based on the pure state  $|\psi\rangle\langle\psi|$ . Above, the direct sum means that we take direct sums of the local Hilbert spaces, which we indicate by the label “ $ij$ ” attached to each local system, whereas  $|\mu\rangle_{ij}$  is the state shared between parties  $i$  and  $j$ .

It is straightforward to check that

$$\rho_I = \frac{1}{|I||I^c|} \bigoplus_{i \in I, j \in I^c} R_{ij} \otimes |ij\rangle\langle ij|^I,$$

hence  $S_\alpha(I) = s + \log(|I||I^c|)$ . On the other hand, if  $J \subset [n]$  with  $J \neq I$ , then there exist  $i \in I$  and  $j \in I^c$  such that either both  $i, j \in J$  or both  $i, j \in J^c$ . Thus,  $\rho_J$  has a direct sum component  $|\mu\rangle\langle\mu|$  and as a consequence an eigenvalue  $\geq \frac{1}{|I||I^c|}$ , hence

$$S_\alpha(J) \leq \frac{1}{1-\alpha} \log \left( \frac{1}{|I||I^c|} \right)^\alpha = C,$$

and we are done.  $\square$



**Theorem 7** For every element  $v \in \mathbb{R}_{\geq 0}^{\mathcal{P}_0[n]}$  there is a vector  $v + O(1) \in \Sigma_\alpha^n$ . To be precise, there exists a constant  $C$ , which may be chosen as  $C = \frac{1}{1-\frac{1}{\alpha}}(\log(n+1))2^{n+1}$ , and states with

$$|S_\alpha(\rho_I) - v_I| \leq C \text{ for all } I \subset [n].$$

In other words, there are no non-trivial homogeneous inequalities constraining the Rényi entropies (with fixed  $\alpha > 1$ ) of a multi-party state: The only restriction is non-negativity:

$$\overline{\mathbb{R}_{\geq 0} \Sigma_\alpha^n} = \mathbb{R}_{\geq 0}^{\mathcal{P}_0[n]}.$$

**Proof** Using proposition 6 this is trivial:  $v = \sum_{I \in \mathcal{P}_0[n]} v_I \delta_I$ , and for each subset  $I \subset [n]$  we can find an  $n$ -party state  $\rho^{(I)}$  such that its entropies approximate  $v_I \delta_I$ , i.e.

$$|S_\alpha(\rho_J^{(I)}) - v_I \delta_{I,J}| \leq \frac{1}{1 - \frac{1}{\alpha}} \log n \text{ for all } I, J \subset [n].$$

Letting  $\rho := \bigotimes_{I \in \mathcal{P}_0[n]} \rho^{(I)}$ , we are done.  $\square$

**Remark 8** From theorem 7 we can see that  $\mathbb{R}_{\geq 0} \Sigma_\alpha^n$  is not a closed set (assuming  $n \geq 2$ ). This is argued in the same way as in remark 4 for the case  $\alpha < 1$ .

The above theorem 7 looks quite similar to theorem 3 for  $0 < \alpha < 1$ . However, whereas there we could conclude that there are no nontrivial inequalities whatsoever for the Rényi entropies of a multi-party state, here we only get that there cannot be any further *homogeneous* inequalities apart from non-negativity.

That this is the most we can hope to obtain follows from the observation that there are other, non-linear and non-homogeneous, inequalities constraining the entropy vectors. In fact,  $\overline{\Sigma_\alpha^n}$  is not a cone at all for  $\alpha > 1$ !

An example of such an inequality was presented by Audenaert [4]: The  $\alpha$ -Schatten norm  $\|\rho\|_\alpha = (\text{Tr} |\rho|^\alpha)^{1/\alpha}$  (the operator norm  $\|\rho\|_\infty = \|\rho\|$  is obtained in the limit  $\alpha \rightarrow \infty$ ) is related to the  $\alpha$ -entropy by

$$S_\alpha(\rho) = \frac{\alpha}{1-\alpha} \log \|\rho\|_\alpha, \quad S_\infty(\rho) = -\log \|\rho\|_\infty,$$

and satisfies

$$\|\rho_A\|_\alpha + \|\rho_B\|_\alpha \leq 1 + \|\rho_{AB}\|_\alpha$$

for arbitrary bipartite state  $\rho_{AB}$ .

The following is a strengthening of Audenaert's inequality.

**Proposition 9** Let  $\rho_{AB}$  be a bipartite state and  $1 < \alpha \leq \infty$ . Define

$$M_\alpha := \max \left\{ (\|\rho_A\|_\infty / \|\rho_A\|_\alpha)^{\alpha-1}, (\|\rho_B\|_\infty / \|\rho_B\|_\alpha)^{\alpha-1} \right\} \text{ for } \alpha < \infty,$$

as well as  $M_\infty := \lim_{\alpha \rightarrow +\infty} M_\alpha = \max\{1/m_A, 1/m_B\}$ , where  $m_A$  and  $m_B$  are the multiplicity of  $\|\rho_A\|_\infty$  and  $\|\rho_B\|_\infty$  as an eigenvalue of  $\rho_A$  and  $\rho_B$ , respectively.

Then,

$$\|\rho_A\|_\alpha + \|\rho_B\|_\alpha \leq \min \left\{ \kappa + \frac{1}{\kappa} \|\rho_{AB}\|_\alpha : M_\alpha \leq \kappa \right\} \quad (4)$$

$$= \begin{cases} 2\sqrt{\|\rho_{AB}\|_\alpha} & \text{if } M_\alpha \leq \sqrt{\|\rho_{AB}\|_\alpha} \\ M_\alpha + \frac{1}{M_\alpha} \|\rho_{AB}\|_\alpha & \text{if } \sqrt{\|\rho_{AB}\|_\alpha} \leq M_\alpha \end{cases} \quad (5)$$

$$\leq 1 + \|\rho_{AB}\|_\alpha. \quad (6)$$

The last inequality holds with equality if and only if at least one of  $\rho_A, \rho_B$  or  $\rho_{AB}$  is a pure state. Moreover, we have  $\|\rho_A\|_\alpha + \|\rho_B\|_\alpha = 1 + \|\rho_{AB}\|_\alpha$  if and only if  $\rho_A$  or  $\rho_B$  is a pure state.

**Proof** We follow Audenaert's proof [4] with a slight modification. Let  $\rho_A = \sum_i \lambda_i |e_i\rangle\langle e_i|$  and  $\rho_B = \sum_j \eta_j |f_j\rangle\langle f_j|$  be eigen-decompositions such that the  $\lambda$ 's and the  $\eta$ 's are arranged in a decreasing order. For  $\alpha = \infty$ , define  $X := \sum_{i=1}^{m_A} \frac{1}{m_A} |e_i\rangle\langle e_i|$ ,  $Y := \sum_{j=1}^{m_B} \frac{1}{m_B} |f_j\rangle\langle f_j|$ , and let  $\beta := 1$ . For  $\alpha < \infty$ , define  $X := \sum_i x_i |e_i\rangle\langle e_i|$  and  $Y := \sum_j y_j |f_j\rangle\langle f_j|$ , where

$$x_i := \lambda_i^{\alpha-1} / \|\rho_A\|_\alpha^{\alpha-1} \quad \text{and} \quad y_j := \eta_j^{\alpha-1} / \|\rho_B\|_\alpha^{\alpha-1},$$

and let  $\beta$  be such that  $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ . Let  $x$  and  $y$  be the vectors formed of the  $x_i$ 's and  $y_j$ 's, respectively. Then  $\|X\|_\beta = \|Y\|_\beta = \|x\|_\beta = \|y\|_\beta = 1$  and  $\|\rho_A\|_\alpha = \text{Tr } X\rho_A$  and  $\|\rho_B\|_\alpha = \text{Tr } Y\rho_B$ . Hence we have, for any real number  $\kappa$ , that

$$\begin{aligned} \|\rho_A\|_\alpha + \|\rho_B\|_\alpha &= \text{Tr}(X \otimes I_B + I_A \otimes Y)\rho_{AB} \\ &= \kappa + \text{Tr}(X \otimes I_B + I_A \otimes Y - \kappa I_A \otimes I_B)\rho_{AB} \\ &\leq \kappa + \text{Tr}(X \otimes I_B + I_A \otimes Y - \kappa I_A \otimes I_B)_+ \rho_{AB} \\ &= \kappa + \text{Tr } Z_\kappa \rho_{AB}, \end{aligned} \quad (7)$$

where  $Z_\kappa := (X \otimes I_B + I_A \otimes Y - \kappa I_A \otimes I_B)_+$  is the positive part.

Consider now the function  $a \mapsto f_\kappa(a) := \left( \sum_j (y_j + a - \kappa)_+^\beta \right)^{\frac{1}{\beta}} = \|(y + a - \kappa)_+\|_\beta$ . This function is convex,  $f_\kappa(\kappa) = \|y\|_\beta = 1$ , and  $f_\kappa(0) = 0$  if we assume that  $\kappa \geq \max_j y_j = \|y\|_\infty$ . Hence, under this assumption,  $f_\kappa(a) \leq a/\kappa$  for every  $0 \leq a \leq \kappa$ . Thus, if  $\kappa \geq \|x\|_\infty$  then

$$\begin{aligned} \|Z_\kappa\|_\beta^\beta &= \|(X \otimes I_B + I_A \otimes Y - \kappa I_A \otimes I_B)_+\|_\beta^\beta = \sum_{i,j} (x_i + y_j - \kappa)_+^\beta = \sum_i f_\kappa(x_i)^\beta \\ &\leq \sum_i (x_i/\kappa)^\beta = \|x\|_\beta^\beta / \kappa^\beta = 1/\kappa^\beta, \end{aligned}$$

i.e.,  $\|Z_\kappa\|_\beta \leq 1/\kappa$ . Due to Hölder's inequality,  $\text{Tr } Z_\kappa \rho_{AB} \leq \|Z_\kappa\|_\beta \|\rho_{AB}\|_\alpha \leq \|\rho_{AB}\|_\alpha / \kappa$ . Combined with (7), this yields

$$\|\rho_A\|_\alpha + \|\rho_B\|_\alpha \leq \kappa + \frac{1}{\kappa} \|\rho_{AB}\|_\alpha =: g(\kappa). \quad (8)$$

Since this is true for every  $\kappa \geq \max\{\|x\|_\infty, \|y\|_\infty\} = M_\alpha$ , we have proved eq. (4).

It is easy to see that  $g$  is strictly convex and it has a global minimum at  $\sqrt{\|\rho_{AB}\|_\alpha} \leq 1$  with minimum value of  $2\sqrt{\|\rho_{AB}\|_\alpha}$ . In particular,  $g$  is strictly decreasing on the interval  $(0, \sqrt{\|\rho_{AB}\|_\alpha}]$  and strictly increasing on  $[\sqrt{\|\rho_{AB}\|_\alpha}, 1]$ , and hence we obtain eq. (5). The inequality (6) is obvious.

By the above properties of  $g$ , we have equality in (6) if and only if  $\max\{M_\alpha, \sqrt{\|\rho_{AB}\|_\alpha}\} = 1$ . Obviously,  $\sqrt{\|\rho_{AB}\|_\alpha} = 1$  if and only if  $\rho_{AB}$  is a pure state, and it is easy to see that  $M_\alpha = 1$  if and only if  $\rho_A$  or  $\rho_B$  is pure.

If  $\rho_A$  is a pure state then  $\rho_{AB} = \rho_A \otimes \rho_B$  and  $1 + \|\rho_{AB}\|_\alpha = 1 + \|\rho_A\|_\alpha \|\rho_B\|_\alpha = 1 + \|\rho_B\|_\alpha = \|\rho_A\|_\alpha + \|\rho_B\|_\alpha$ , and a completely similar argument works if  $\rho_B$  is pure. On the other hand, if  $\|\rho_A\|_\alpha + \|\rho_B\|_\alpha = 1 + \|\rho_{AB}\|_\alpha$  then equality has to hold in (6), and hence  $\rho_A, \rho_B$  or  $\rho_{AB}$  has to be pure. If  $\rho_{AB}$  is pure but  $\rho_A$  is not then  $\|\rho_A\|_\alpha = \|\rho_B\|_\alpha < 1$  and hence  $\|\rho_A\|_\alpha + \|\rho_B\|_\alpha < 2 = 1 + \|\rho_{AB}\|_\alpha$ . This proves the last assertion about the equality case.  $\square$

It appears that even for two parties, no description of  $\Sigma_\alpha^2$  or  $\overline{\Sigma_\alpha^2}$  is known. Nor, which other inequalities there are constraining the latter.

## V. CLASSICAL CASE

As remarked in the introduction, if restricted to classical states  $\rho$ , the Rényi entropies are monotonic, i.e.

$$I \subset J \Rightarrow S_\alpha(\rho_I) \leq S_\alpha(\rho_J). \quad (9)$$

(More generally, this holds for separable states, thanks to the majorisation result of Kempe and Nielsen [33].) In this section, we denote the set of  $\alpha$ -entropic vectors of a generic distribution of  $n$  random variables by  $\Gamma_\alpha^n$ . In other words, this is a subset of  $\Sigma_\alpha^n$ , with the restriction that the underlying states are classical.

The extremal rays of the convex cone  $\mathcal{MO}^n$  described by non-negativity and eqs. (9) – which thus contains  $\Gamma_\alpha^n$  – are easy to describe in combinatorial language [19]: They are precisely the rays spanned by the indicator functions

$$i_{\mathcal{U}} : I \mapsto \begin{cases} 1 & \text{if } I \in \mathcal{U}, \\ 0 & \text{otherwise.} \end{cases}$$

of a nonempty set family  $\mathcal{U} \subset \mathcal{P}_0[n]$  with the property that  $J \supset I \in \mathcal{U}$  implies  $J \in \mathcal{U}$  (hence always  $[n] \in \mathcal{U}$ ). Such set families are known in combinatorics as “upsets” (or sometimes “ideals”).

Some of the simplest upset are generated by a single element:

$$\uparrow J = \{I \in \mathcal{P}_0[n] : J \subset I\}.$$

These have the property that the unique minimal element of the family is  $J$ . Note also that an upset contains, with each element  $J$ , the entire  $\uparrow J$ . This means that every upset  $\mathcal{U}$  can be written

$$\mathcal{U} = \bigcup_{J \in \mathcal{L}} \uparrow J,$$

with  $\mathcal{L}$  the set of minimal elements of  $\mathcal{U}$ .

For instance for  $n = 2$ , there are four upsets and clearly all four associated rays are attainable (whole ray for  $\alpha < 1$ , sufficiently long dilution for  $\alpha > 1$ ).

Next we show that this is the only difference to the quantum case, at least as long as we are only looking for homogeneous inequalities. Namely, the only homogeneous inequalities obeyed by the classical  $\alpha$ -entropies are non-negativity and monotonicity.

**Theorem 10** *Let  $0 < \alpha < 1$ . For any upset  $\mathcal{U} \subset \mathcal{P}_0[n]$  and all  $s > 0$ , there is a vector  $s i_{\mathcal{U}} + O(1) \in \Gamma_\alpha^n$ . To be precise, there exists a probability distribution  $P$  with*

$$s \leq H_\alpha(P_I) \leq s + \frac{\log |\mathcal{L}| + 1}{1 - \alpha} \quad \text{for } I \in \mathcal{U}, \quad H_\alpha(P_I) \leq \log |\mathcal{L}| + 1 \quad \text{for } I \notin \mathcal{U}.$$

*In particular, for  $s \rightarrow \infty$  we obtain  $i_{\mathcal{U}} \in \overline{\mathbb{R}_{\geq 0} \Gamma_\alpha^n}$ . As a consequence,  $\overline{\mathbb{R}_{\geq 0} \Gamma_\alpha^n} = \mathcal{MO}^n$ .*

**Proof** Let  $\mathcal{U} = \bigcup_{J \in \mathcal{L}} \uparrow J$ , which can be achieved by choosing  $\mathcal{L}$  to be the minimal elements of  $\mathcal{U}$ . For each  $i \in [n]$ , let the local alphabet  $\mathcal{X}_i$  be of the form

$$\mathcal{X}_i = \cup_{J \in \mathcal{L}}^* \mathcal{X}_{i,J}, \text{ where } \begin{cases} \mathcal{X}_{i,J} = \{0\} \cup [M_i], & \text{if } i \in J, \\ |\mathcal{X}_{i,J}| = 1 & \text{if } i \notin J, \end{cases}$$

where  $\cup^*$  denotes disjoint union, i.e.,  $\mathcal{X}_{i,J}$  and  $\mathcal{X}_{i,J'}$  are disjoint if  $J \neq J'$ . For each  $J \in \mathcal{L}$ , let  $P_{t_J; \{M_i: i \in J\}}$  be a probability distribution on  $\times_{i \in J} \mathcal{X}_{i,J} \subset \times_{i \in J} \mathcal{X}_i$ , defined as in lemma 1, with  $t_J^\alpha M_J^{1-\alpha} = 2^{s'(1-\alpha)}$ , where  $s' := s + \frac{\alpha}{1-\alpha} \log |\mathcal{L}|$ . Let  $Q_J := P_{t_J; \{M_i: i \in J\}} \otimes \delta_{J^c}$ , where  $\delta_{J^c}$  is the trivial probability distribution on the single-element set  $\times_{i \in J^c} \mathcal{X}_{i,J}$ . Note that the supports of  $Q_J$  and  $Q_{J'}$  are disjoint for  $J \neq J'$ . We claim that

$$P := \frac{1}{|\mathcal{L}|} \sum_{J \in \mathcal{L}} Q_J = \frac{1}{|\mathcal{L}|} \sum_{J \in \mathcal{L}} P_{t_J; \{M_i: i \in J\}} \otimes \delta_{J^c}$$

has the desired properties for large enough  $M_0 := \min_i M_i$ .

Indeed, it is easy to see that for any  $\emptyset \neq I \subset [n]$ ,

$$\begin{aligned} H_\alpha(P_I) &= \frac{1}{1-\alpha} \log \left( \frac{1}{|\mathcal{L}|^\alpha} \left( \sum_{I \cap J = \emptyset} 1 + \sum_{J \subset I} ((1-t_J)^\alpha + 2^{s(1-\alpha)}) + \sum_{\emptyset \neq J \setminus I \subsetneq J} ((1-t_J)^\alpha + 2^{s(1-\alpha)}/M_{J \setminus I}^{1-\alpha}) \right) \right) \\ &= \frac{1}{1-\alpha} \log \left( \frac{1}{|\mathcal{L}|^\alpha} \left( \sum_{I \cap J = \emptyset} 1 + \sum_{I \cap J \neq \emptyset} (1-t_J)^\alpha \right) + \frac{1}{|\mathcal{L}|^\alpha} \sum_{J \subset I} 2^{s(1-\alpha)} + \frac{1}{|\mathcal{L}|^\alpha} \sum_{\emptyset \neq J \setminus I \subsetneq J} 2^{s(1-\alpha)}/M_{J \setminus I}^{1-\alpha} \right) \\ &\xrightarrow{M_0 \rightarrow +\infty} \frac{1}{1-\alpha} \log \left( |\mathcal{L}|^{1-\alpha} + 2^{s'(1-\alpha)} \frac{\#\{J \in \mathcal{L} : J \subset I\}}{|\mathcal{L}|^\alpha} \right), \\ &= \frac{1}{1-\alpha} \log \left( |\mathcal{L}|^{1-\alpha} + 2^{s(1-\alpha)} \#\{J \in \mathcal{L} : J \subset I\} \right), \end{aligned} \tag{10}$$

where in all summations over  $J$  above,  $J \in \mathcal{L}$  is implicit. Now, if  $I \in \mathcal{U}$  then there exists a  $J \in \mathcal{L}$  such that  $J \subset I$ , and hence the expression in (10) can be lower bounded by  $s$ , and upper bounded by  $\frac{1}{1-\alpha} \log \left( |\mathcal{L}| (1 + 2^{s(1-\alpha)}) \right) < s + \frac{\log |\mathcal{L}| + 1}{1-\alpha}$ . On the other hand, if  $I \notin \mathcal{U}$  then  $\#\{J \in \mathcal{L} : J \subset I\} = 0$  and hence the expression in (10) is equal to  $\log |\mathcal{L}|$ .  $\square$

**Theorem 11** Let  $1 < \alpha \leq \infty$ . For any upset  $\mathcal{U} \subset \mathcal{P}_\emptyset[n]$  and  $s > 0$ , there is a vector  $s i_{\mathcal{U}} + O(1) \in \Gamma_\alpha^n$ . To be precise, there exists a constant  $C$ , which may be chosen as  $C = \frac{1}{1-\frac{1}{\alpha}} \log(2n^k)$  if  $\mathcal{U}$  is generated by  $k$  elements ( $k < 2^n$  always), and a probability distribution  $Q$  with

$$s \leq H_\alpha(Q_I) \leq s + C \text{ for } I \in \mathcal{U}, \quad H_\alpha(Q_I) \leq C \text{ for } I \notin \mathcal{U}.$$

In particular, for  $s \rightarrow \infty$  we obtain  $i_{\mathcal{U}} \in \overline{\mathbb{R}_{\geq 0} \Gamma_\alpha^n}$ . As a consequence,  $\overline{\mathbb{R}_{\geq 0} \Gamma_\alpha^n} = \mathcal{MO}^n$ .

**Proof** Let  $M$  be the smallest natural number such that  $s \leq 1 + \log M$ . We take as our building blocks the distributions  $Q_{[n]:M}$  from lemma 5 and its proof, for simplicity with uniform  $R$  on  $[M]$ . Furthermore, define the following uniform distribution on the diagonal of  $[M]^n$ :

$$\Delta_M := \frac{1}{M} \sum_{x=1}^M \delta_x \otimes \delta_x \otimes \cdots \otimes \delta_x.$$

Now, for upset  $\mathcal{U} = \bigcup_{J \in \mathcal{L}} \uparrow J$ , let

$$Q := \frac{1}{2} \Delta_M \oplus \frac{1}{2} \bigotimes_{J \in \mathcal{L}} (Q_{J:M} \otimes \delta_{J^c}),$$

where  $\delta_{J^c}$  as before refers to a generic point mass for parties  $J^c = [n] \setminus J$ , the product over  $J \in \mathcal{L}$  implies Cartesian products of the local alphabets, and the direct sum likewise a direct sum of the *local* alphabets.

The first term in the direct sum makes sure that in each marginal the largest probability value occurring is at least  $\geq \frac{1}{2M}$ , with multiplicity  $M$ , not allowing the Rényi entropy of any subset to become larger than  $\log M + \frac{1}{1-\alpha}$ . Turning to the second term, note that in the tensor product over  $J \in \mathcal{L}$ , the distributions are designed such that for  $J \subset I$ , the distribution  $(Q_{J:M} \otimes \delta_{J^c})_I$  is uniform on  $|J|M$  elements, whereas for  $J \not\subset I$ , it has at least one value  $\geq \frac{1}{|J|}$ .

Thus,

$$H_\alpha(Q_I) \begin{cases} \leq \frac{1}{1-\alpha} \log(2n^k) & \text{for } I \notin \mathcal{U}, \\ \geq 1 + \log M & \text{for } I \in \mathcal{U}, \end{cases}$$

and we are done.  $\square$

## VI. EPILOGUE

We have carried out an analysis of the inequalities obeyed by quantum Rényi entropies in multi-partite systems, in analogy to the very deep ongoing programme for the von Neumann entropy. In the quantum case, our findings can be summarized concisely as saying that apart from trivial non-negativity of individual entropies there are no inequalities obeyed by the Rényi  $\alpha$ -entropies of a multipartite state, when  $0 < \alpha < 1$ . For  $1 < \alpha \leq \infty$  there are no other homogeneous inequalities, but the set of attainable entropic vectors is not a cone, meaning that there are further, non-homogeneous inequalities. In the classical case (and more broadly that of separable quantum states) there is furthermore monotonicity in the sense that a smaller subset of parties cannot have larger entropy, and we could show similarly that this is the only homogeneous inequality for all  $\alpha \neq 1$ . It is curious to contrast this with the limit  $\alpha = 1$ , the von Neumann entropy, which is subject to subadditivity and strong subadditivity, as well as triangle inequality and weak monotonicity, all crucial relations for the development of statistical mechanics and quantum information theory. The classical case has even more inequalities, due to Zhang and Yeung and subsequent work.

We did not discuss the other limit  $\alpha = 0$ , for which the Rényi entropy is the logarithm of the rank of the density operator, which indeed behaves rather differently from the other  $\alpha$ -entropies: For one thing, it takes only discrete values in the logarithm of integers, and it is discontinuous. Furthermore, it is easy to see that it obeys subadditivity

$$S_0(\rho_{I \cup J}) \leq S_0(\rho_I) + S_0(\rho_J),$$

and it is unknown which other inequalities (whether homogeneous, linear or other) it satisfies. Note however that it definitely does not satisfy strong subadditivity [9].

We leave a few other open questions and further directions for future investigations: For instance, we would like to know all necessary non-homogeneous inequalities to describe the  $\overline{\Sigma}_\alpha^n$ . Note that the classical/separable sets  $\Gamma_\alpha^n$  for  $\alpha > 1$  are not cones either, but what about  $0 < \alpha < 1$ ? Finally, and perhaps most interestingly in the light of recovering what rich structure is known for  $\alpha = 1$ , can we extend the present investigation to relations between Rényi entropies for different  $\alpha$ ? For example, it is well-known that for  $\alpha < \beta$ ,  $S_\alpha(\rho) \geq S_\beta(\rho)$ , or  $S_\alpha(\rho_{AB}) \leq S_\alpha(\rho_B) + S_0(\rho_B)$ , but we do not know which other inequalities, if any, exist.

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- [1] J. Aczél, Z. Daróczy, *On Measures of Information and Their Characterizations*, Academic Press, New York, 1975.
  - [2] R. Alicki, M. Fannes, *Quantum dynamical systems*, Oxford University Press, Oxford, 2001.
  - [3] G. Aubrun, I. Nechita, “Catalytic Majorization and  $\ell_p$  Norms”, *Commun. Math. Phys.* **278**:133-144 (2008).
  - [4] K.M.R. Audenaert, “Subadditivity of  $q$ -entropies for  $q > 1$ ”, *J. Math. Phys.* **48**:083507 (2007); arXiv[math-ph]:0705.1276.
  - [5] K.M.R. Audenaert, M. Nussbaum, A. Szkoła, F. Verstraete, “Asymptotic error rates in quantum hypothesis testing”, *Comm. Math. Phys.* **279**:251–283 (2008).
  - [6] C.H. Bennett, H.J. Bernstein, S. Popescu, B. Schumacher, “Concentrating partial entanglement by local operations”, *Phys. Rev. A* **53**(4):2046–2052 (1996).
  - [7] I. Bjelakovic, A. Szkoła, “The Data Compression Theorem for Ergodic Quantum Information Sources”, *Quantum Information Processing* **4**:49–63 (2005).
  - [8] J. Cadney, N. Linden, A. Winter, “Infinitely many constrained inequalities for the von Neumann entropy”, *IEEE Trans. Inf. Theory* **58**(6):3657–3663 (2012); arXiv:1107.0624.
  - [9] J. Cadney, personal communication, June 2012.
  - [10] I. Csiszár, “Generalized cutoff rates and Rényi’s information measures”, *IEEE Trans. Inf. Theory* **41**:26–34, (1995).
  - [11] R. Dougherty, C. Freiling, K. Zeger, “Six new non-Shannon information inequalities”, *IEEE Int. Symp. Inf. Theory*, pp. 233-236 (2006); “Networks, matroids, and non-Shannon information inequalities”, *IEEE Trans. Inf. Theory* **53**(6):1949–1969 (2007).
  - [12] J. Eisert, M. Cramer, M.B. Plenio, “Area laws for the entanglement entropy - a review” *Rev. Mod. Phys.* **82**:277 (2010).
  - [13] L. Hardy, “Method of areas for manipulating the entanglement properties of one copy of a two-particle pure entangled state”, *Phys. Rev. A* **60**(3):1912-1923 (1999).
  - [14] P. Hayden, A. Winter, “Communication cost of entanglement transformations”, *Phys. Rev. A* **67**:012326 (2003).
  - [15] S.M. Giampaolo, S. Montangero, F. Dell’Anno, S. De Siena, F. Illuminati, “Scaling of the Rényi entropies in gapped quantum spin systems: Entanglement-driven order beyond symmetry breaking”, arXiv:1208.0735 (2012).
  - [16] M. Hayashi, *Quantum Information: An Introduction*, Springer Verlag, Berlin Heidelberg, 2006.
  - [17] M. Hayashi, “Error exponent in asymmetric quantum hypothesis testing and its application to classical-quantum channel coding”, *Phys. Rev. A* **76**:062301 (2007).
  - [18] F. Hiai, M. Mosonyi, T. Ogawa, “Error exponents in hypothesis testing for correlated states on a spin chain”, *J. Math. Phys.* **49**:032112 (2008).
  - [19] B. Ibinson, N. Linden, A. Winter, “All Inequalities for the Relative Entropy”, *Commun. Math. Phys.* **269**(1):223–238 (2007).
  - [20] E.H. Lieb, M.B. Ruskai, “Proof of the strong subadditivity of quantum-mechanical entropy”, *J. Math. Phys.* **14**(12):1938–1941 (1973).

- [21] N. Linden, A. Winter, "A new inequality for the von Neumann entropy", *Commun. Math. Phys.* **259**:129-138 (2005).
- [22] R. Koenig, S. Wehner, "A strong converse for classical channel coding using entangled inputs", *Phys. Rev. Lett.* **103**:070504 (2009).
- [23] K. Makarychev, Y. Makarychev, A. Romashchenko, N. Vereshchagin, "A new class of non-Shannon-type inequalities for entropies", *Commun. Inf. Syst.* **2**(2):147-165 (2002).
- [24] F. Matúš, "Infinitely many information inequalities", *IEEE Int. Symp. Inf. Theory*, pp. 41-44 (2007).
- [25] A. Montanaro, A. Winter, "A Lower Bound on Entanglement-Assisted Quantum Communication Complexity", in: *Proc. ICALP 2007*, L. Arge *et al.* (eds.), LNCS 4596, Springer Verlag, Berlin Heidelberg, 2007.
- [26] M. Mosonyi, F. Hiai, T. Ogawa, M. Fannes, "Asymptotic distinguishability measures for shift-invariant quasi-free states of fermionic lattice systems", *J. Math. Phys.* **49**:072104 (2008).
- [27] M. Mosonyi, "Hypothesis testing for Gaussian states on bosonic lattices", *J. Math. Phys.* **50**:032105 (2009).
- [28] M. Mosonyi, N. Datta, "Generalized relative entropies and the capacity of classical-quantum channels", *J. Math. Phys.* **50**:072104 (2009).
- [29] M. Mosonyi, F. Hiai, "On the quantum Rényi relative entropies and related capacity formulas", *IEEE Trans. Inf. Theory*, **57**:2474-2487 (2011).
- [30] H. Nagaoka, "The converse part of the theorem for quantum Hoeffding bound", *arXiv:quant-ph/0611289* (2006).
- [31] J. von Neumann, "Thermodynamik quantenmechanischer Gesamtheiten", *Nachr. Ges. Wissenschaften Göttingen*, year 1927, pp.273-291 (1927),
- [32] M.A. Nielsen, "Conditions for a Class of Entanglement Transformations", *Phys. Rev. Lett.* **83**(2):436-439 (1999).
- [33] M.A. Nielsen, J. Kempe, "Separable States Are More Disordered Globally than Locally", *Phys. Rev. Lett.* **86**(22):5184-5187 (2001).
- [34] T. Ogawa, H. Nagaoka, "Strong converse to the quantum channel coding theorem", *IEEE Trans. Inform. Theory* **45**:2428-2433 (1999).
- [35] T. Ogawa, H. Nagaoka, "Strong converse and Stein's lemma in quantum hypothesis testing", *IEEE Trans. Inform. Theory* **47**:2428-2433 (2000).
- [36] N. Pippenger, "What are the laws of information theory?", 1986 Special Problems in Communication and Computation Conference, Palo Alto, CA, 3-5 September 1986.
- [37] N. Pippenger, "The inequalities of quantum information theory", *IEEE Trans. Inf. Theory* **49**(4):773-789 (2003).
- [38] A. Rényi, "On Measures of Entropy and Information", in: *Proc. Fourth Berkeley Symp. on Math. Statist. and Prob.*, J. Neyman (Ed.), pp. 547-561 (1960).
- [39] C.E. Shannon, *Bell Syst. Tech. J.* **27**:379-423 & 623-656 (1948).
- [40] C. Tsallis, *J. Stat. Phys.* **52**(1/2):479-487 (1988).
- [41] G. Vidal, J.I. Latorre, E. Rico, A. Kitaev, "Entanglement in quantum critical phenomena" *Phys. Rev. Lett.* **90**:227902 (2003).
- [42] W. van Dam, P. Hayden, "Renyi-entropic bounds on quantum communication", *arXiv:quant-ph/0204093* (2002).
- [43] M.M. Wilde, "From Classical to Quantum Shannon Theory", *arXiv:1106.1445* (2011).
- [44] R.W. Yeung, "A Framework for Linear Information Inequalities", *IEEE Trans. Inf. Theory* **43**(6):1924-1934 (1997).
- [45] Z. Zhang, R.W. Yeung, "A Non-Shannon Type Conditional Inequality of Information Quantities", *IEEE Trans. Inf. Theory* **43**(6):1982-1985 (1997).
- [46] Z. Zhang, R.W. Yeung, "On the characterization of entropy function via information inequalities", *IEEE Trans. Inf. Theory* **44**(4):1440-1452 (1998).